“How do I prove it?”: A Narrative Inquiry into Advancing a Child’s School Mathematics

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Abstract: This article celebrates Chen, a Grade 3 student who shows his lived curriculum in the context of his school. This narrative inquiry is grounded in student-centred curriculum making. We explore Chen’s inherent curiosity about developing formal techniques to prove mathematical statements—a standard process performed by mathematicians. We share observations and transcribed excerpts of video conversations that demonstrate Chen’s remarkable understanding of the following results: the role examples play in proving and disproving statements; the application of direct proofs; the structure of proofs by contradiction; and the structure of proofs by induction. Stavros’s approach to Chen’s math education has implications in how students transition from routine calculations to generalized and abstracted ideas to solving problems by agreeing upon a shared understanding of proof structures. We conclude with recommendations on *planning thoughts* and *action thoughts* as strategies to create educative experiences that support both the learning and identity making of children.

Keywords: narrative inquiry, gifted students, school mathematics, curriculum making, mathematical proofs

Introduction

“How Do I Prove It?”

My university Zoom lecture ran overtime, so I quickly switched to Microsoft Teams where Chen—a precocious eight-year-old in Grade 3—was waiting patiently. His face was mostly hidden behind a blue mask. This is our third year working together, and our second year navigating the pandemic. K-12 students have returned to school under strict physical distancing and masking guidelines. As a non-employee, I’m not allowed in the school, so I arranged with Chen’s teacher to meet with him virtually.

Chen’s eyes light up when he sees me. He’s smiling beneath the mask. Scattered across my desk are notes of planned thoughts *for activities that may or may not be used, depending on where his playful curiosity takes us. I share my screen and show him Goldbach’s Conjecture—one of the oldest and most infamous open problems in number theory. It proposes that every even integer greater than 2 is the sum of two prime numbers. It was Chen’s idea to explore this last week, and I’d forgotten how captivating these problems can be. I ended up spending hours lost in it. Chen reminds me that playfulness is important.*

Last week, Chen began testing the conjecture himself:

$4 = 2 + 2;\ 6 = 3 + 3;\ 8 = 3 + 5;\ 10 = 5 + 5;\ 12 = 5 + 7; \ldots$

I tell him about the largest number that’s been tested, some heuristic results, and other interesting facts about the conjecture’s history. He listens intently, calculating as we go. Then he looks up and asks, “All the examples I tried work so far. How do I prove it?” (Field text, October 2021).

Greetings, tansi, taanishi, bonjour. I am Stavros, working in teacher education and conducting educational research in the College of Education at University of Saskatchewan, located on Treaty 6 territory and homeland of the Métis. At the time of this project, I was collaborating with teachers and students in their mathematics classrooms. My primary focus was working with Indigenous teachers and students. I took on this project with Chen in addition to my regular responsibilities, ensuring that it did not compromise the time or financial resources allocated to my existing programming.

I am Shaun, also based in the College of Education at University of Saskatchewan on Treaty 6 territory and homeland of the Métis. My background includes working as a primary school teacher and later as a professor teaching elementary mathematics methods. I joined Stavros in this project to bring together my expertise in teacher education with his academic background in mathematics.

Our professional work with children and youth is grounded in Connelly and Clandinin's (1988) concept of curriculum making. While 'curriculum' often refers to the mandated documents teachers use to plan lessons, Aoki (1993) encouraged researchers to consider the diverse *lived* curricula that emerge through student-teacher interactions. Working with Chen reminded us of the limitations of the mandated curriculum, which did not adequately support his educational needs or experiences. It was through Chen's enthusiasm and playful engagement with mathematics that we were able to discern the direction he wished to pursue. Centering curriculum making shaped both our research approach and our pedagogical stance to teaching mathematics.

In previous work (Stavrou & Murphy, 2024), we illustrated the experiences coming to meet Chen and the complexities around his lived curriculum as a strong mathematics student. He sometimes described his mathematics class as boring and lacking opportunities to build on what he already knew or was curious about. This sentiment was echoed by his teachers, who noted that he quickly completed assigned work, had prior exposure to concepts beyond the curriculum, asked questions that exceeded the teacher's content knowledge, and sought alternative methods for solving problems. Continuing in this direction, our study offers an example of how curriculum evolves in real time within the context of mathematics learning. Chen inspired us to design activities that allowed him to engage with mathematical processes that genuinely interested him—something that is difficult to achieve within the constraints of a mandated curriculum, especially in classrooms with diverse learning needs.

One teacher asked, "What do we do with him?" Her question points to the imperative of providing differentiated learning and additional supports to diverse students, as well as the challenges inherent to that. In this article, we share experiences that highlight Chen's spontaneous interest in learning and applying mathematical proofs—a formal logical structure central to mathematical problem-solving. Stavros's field texts illustrate Chen's grasp of key proof strategies, including the role of examples in proving and disproving statements, direct proofs, proofs by contradiction, and proofs by induction. A recurring question Chen posed that guided the trajectory of our sessions was: “How do I prove it?”

The remainder of this article is organized as follows. First, we situate ourselves within the broader research landscape by describing mathematical proofs and examining the role of teachers in curriculum making and identity formation for children with exceptional needs. Next, we outline our methodological approach of using narrative inquiry (NI) to explore the lived experiences of Chen, as well as our own. We then discuss our findings through field notes that narrate our sessions, illustrating an approach to Chen’s relational curriculum making and demonstrating how students move from routine calculations to abstracted thinking. The transition to abstraction is facilitated through a shared understanding of proof structures. Finally, we conclude with reflections on *planning thoughts* and *action thoughts* as strategies educators can use to create educative experiences that support both the learning and identity making of children.

Literature Review

Locating Ourselves in the Broader Research

We do not provide a comprehensive overview of the design and characteristics of differentiated tasks for students with gifted mathematical abilities, nor is this article concerned with evaluating research on teaching models. Rather, our research is an example of the trend towards qualitative studies demonstrating the value of recognizing, differentiating, and directing mathematically precocious students (Ozdemir & Isiksal Bostan, 2021) with particular attention to a Deweyan (1938) ontology of experience and the ways educative experiences enhance growth.

Proofs in Mathematics Education

The phenomenon of how proofs are perceived, taught, and applied in educational settings to validate mathematical statements and deepen understanding involves students’ conceptual understanding, teachers’ content and pedagogical knowledge, and instructional practices (Faizah et al., 2020; Lesseig et al., 2019). There are challenges in distinguishing proof from justification, opportunities for developing argumentation, and adapting teaching methods to promote meaningful engagement by connecting formal reasoning with intuitive approaches (Herizal et al., 2019; Stavrou, 2014). Hanna and Knipping (2020) surveyed how the role of mathematical proofs in education has changed between 1980 and 2020, tracing its evolution from post-New Math backlash to modern digital tools that help with proof construction. They provided an historical context which emphasized axiomatic rigor but faced backlash in the 1950s-1970s, leading to a 1980s shift toward constructivism, heuristics from Polya and Lakatos, and informal proofs valuing understanding over formality. They noted that from 1980-2000, studies highlighted student proof schemes (e.g., empirical versus deductive) and roles of intuition, while 2000-2020 studies brought sociocultural views, software for conjecture-testing, and debates on the aims of proofs.

Most attention surrounds the difficulties students have understanding proofs (Epp, 2003). Largely due to the advanced nature and prerequisite knowledge of mathematical proofs, most literature describing its teaching and learning is centred around high school and post-secondary students. An exception by Er and Dinc Artut (2022) explored 25 fifth-grade Turkish students who were diagnosed as gifted and were administered tests through Balacheff's taxonomy to determine their levels of proof conceptualization and development.

This article uniquely considers a child in Grade 3. We were not interested in quantifying or qualifying his level of giftedness or producing a diagnosis. Rather, we were interested in identity making and navigating schooling contexts through a NI (Huber et al., 2003) that explored Chen's experiences receiving curricular supports. The successful results of Stavros's approach to Chen's education has implications in understanding how students transition from routine calculations to abstracted ideas required in solving problems by agreeing upon a shared understanding of proof structures. Proofs are not part of Chen's curriculum, and therefore the inclusion of it functions differently in his education.

Role of Educators in Differentiated Mathematics Instructions

In the broader research, Chen's experiences can be described in terms of differentiated mathematics instruction (Freiman, 2011) geared towards a talented student (Rotigel & Fello, 2004) in ways that appropriately challenge and inspire his creative disposition (Boaler, 2015; Feldhusen & Kroll, 1991; Sheffield, 2017) and mathematical interests (Lampen & Brodie, 2020). As educators, our role is a form of cognitive apprenticeship in which we provide extrinsic and intrinsic supports, such as scaffolding and reflective meaning-making (Bouta & Paraskeva, 2012). Using Chen's curiosity, we attended to the necessity of exploring other areas of mathematics when considering a mathematical problem. Stavros's background doing mathematics research in algebra informed his understanding of how to explore branches of mathematics, which he imparted to Chen.

As educators, we must re-imagine ourselves as curriculum makers by considering the lives of our students, as we exemplified in this article. While expectations on teachers exist, lives—not mandated curriculum—need to be the starting point. Our intention is to show and tell the ways a child's school mathematics was re-shaped through his inherent curiosity. We argue that the significance of Chen developing the language and logic of mathematical proofs is that they are fundamental to describing and advancing mathematical phenomena in research and practice.

In terms of identity making, Chen made explicit that he wanted to be able to do mathematics every day like Stavros does. Chen was curious about problems that Stavros researches, and began to see himself as a mathematician who also solves problems. Through the relational curriculum they co-composed, Chen's identity as a *mathematical* person grew as he established the conventional discourse mathematicians use to discuss proofs.

Narrative Inquiry

The research in this article was constructed and conducted using a NI methodology (Clandinin, 2013). The commonplaces of NI—consisting of sociality, temporality, and spatiality—shaped our understanding of the field texts (Connelly & Clandinin, 2006). These commonplaces provided a context through which to view and understand the field texts. Individuals live storied lives on storied landscapes (Clandinin, 2013), and in this research we inquired into a research experience related to the teaching and learning of advanced school mathematics with Chen.

The teaching of mathematics is not only situated in an understanding of subject matter but also resides in issues of identity making. The understanding of curriculum as a course of life (Connelly & Clandinin, 1988) is a way of considering the complexity of curriculum beyond subject matter. The curriculum commonplaces of subject matter, learner, teacher, and milieu (Schwab, 1978) position curriculum as an interactive and dynamic event. This event can be understood as curriculum making, which implies that curriculum is involved in life writing and therefore, identity making (Clandinin et al., 2006). For example, when engaged in mathematics learning, children are also in the process of life-writing and their identity making. The involvement of university instructors providing differentiated learning for Chen is a response to his need for identity making, taking up a complex way of knowing and being in relation to mathematics education. According to Trumbull et al. (2002), "individual students construct an understanding of mathematics concepts on the basis of their experiences within a community" (p. 2). In our professional practice, community includes the involvement of university instructors in Chen's education.

Methods

Stavros met with Chen once a week during his regularly scheduled math class, through Microsoft Teams. The sessions were 40 minutes in length. Teams connected Stavros (working in his home office) with Chen (in a private resource room in an urban elementary school, located in a Western Canadian prairie province.) The timeframe highlighted in this research occurred from September 2021 to December 2021.

The data collected is in the form of field notes of observations made during math sessions with Chen; transcribed summaries of audio-recordings taken during math sessions; video recordings of Teams sessions; and artifacts such as lesson plans, curriculum resources, and university mathematics textbooks.

During their one-on-one sessions, Chen solved problems Stavros prepared ahead of time, but they also engaged in topics that emerged as Chen directed his attention to other problems that piqued his interest (e.g., problems he was given by his teachers or problems he learned at home while watching YouTube videos). The fragments of narrative throughout the article were composed either from Chen's words, the solutions he provided to the problems Stavros posed, or from Stavros's interpretation of Chen's recorded "actions, doing, and happenings, all of which are narrative expressions" (Clandinin & Connelly, 2000, p. 79).

Stavros recorded the Teams sessions during the periods they were held. The purpose of recording the sessions was to summarize interesting discussions to share in this article. His parents and teachers did not ask to view the recordings because Stavros adequately summarized the important details of the overall progress and performance. There were no audio transcripts.

Monthly member checks throughout the school year consisted of conversations with Chen's parents and teachers. They gave Stavros autonomy and authority to direct Chen's learning, which was largely motivated by Chen's engagement and proclivities towards specific topics.

Shaun came alongside Stavros to collaboratively analyze the field notes. As narrative inquirers, we interpret Chen's classroom experiences by engaging with the stories he tells about his schooling within a three-dimensional inquiry space—considering time, social relationships, and place. Shaun listened to Stavros's and Chen's narratives, observing and co-constructing meanings alongside them, integrating personal, social, and cultural contexts (Kim, 2016; Wei, 2023). This process of reconstructing narratives to understand the complexity of Chen's lived classroom experiences involves considering how experiences are shaped by grand narratives (e.g., institutional, communal, and familial), deciding what stories are told, and the relational-ethical aspects of representing lives in the midst (Clandinin & Connelly, 2000; Clandinin et al., 2006).

Setting the Stage for Experiences with Chen

Stavros used *planning thoughts* and *action thoughts* (Clark & Peterson, 1986) to understand and shape his sessions with Chen. Planning thoughts are those that take place before or after lessons, when teachers can reflect or reassess in a quiet environment; whereas action thoughts emerge during classroom interactions, where immediacy and complexity are unavoidable as teachers and students engage with each other (Doyle, 1986).

Stavros engaged in planning thoughts during the time between sessions with Chen, which involved creating math problems with purposeful scaffolding, curating content that allowed Chen to explore topics in breadth and depth, and designing activities with tasks requiring different levels of attention and structure.

Stavros shared with Shaun an instance when he did not have adequate time to prepare between sessions, where Chen said, “Maybe you should come prepared so that I don’t have to wait.” (Field text, May 2019). This comment—made early in their time working together—remained at the back of Stavros’s mind, reminding him that preparedness shapes educative experiences.

Coleman (2014) explained action thoughts are quick, spontaneous, and critical to the successful implementation of a lesson in which there is little time for reflection. For example, if the solution to a math problem did not work out the way Stavros intended—causing a temporary disruption in their session—Chen sometimes made impatient remarks, perceiving the disruption of flow as Stavros’s lack of preparedness, rather than an organic aspect of learning. By having a series of activities planned, Stavros can pivot. For example, as Stavros continued providing planned information on the Goldbach Conjecture, he noticed Chen became fixated on the concept of proving the ‘simple’ statement, which Stavros knew wasn’t likely because of its status as an unsolved problem. Stavros used this opportunity to shift the direction of the session towards techniques of proving mathematical statements.

This inquiry drew our attention to the need for curriculum—understood as life making—to provide educative experiences (Dewey, 1938) for children. Results of this article included Chen’s understanding of the role examples play in proving and disproving statements; the application of direct proofs; the structure of proofs by contradiction; and the structure of proofs by induction.

Findings

Direct Proofs and the Role of Examples: Interruptions in Chen’s Thinking

A *direct proof* is a logical argument where one assumes the truth of the hypothesis and uses previously established facts to derive the conclusion without resorting to contradictions or contrapositions. For statements of the form ‘if P then Q’, a direct proof assumes P to demonstrate Q must be true through a sequence of logical steps. This deductive practice is vital to developing students' mathematical reasoning (Miyazaki et al., 2022).

Stavros asked Chen to prove the statement, 'If x and y are even integers, then $x + y$ is an even integer.' Their conversation is summarized below:

Chen: That's an easy one. Adding two even numbers together gives an even number. Like, 2 plus 2 is 4; 10 plus 12 is 22; 30 plus 300 is 330.

Stavros: You listed three specific examples that work; but that doesn't prove the general statement.

Chen: Why not?

Stavros: For example, if I ask you to prove the statement, 'Dividing two integers creates an integer.' If you give me examples like 2 divided by 2 is 1 or 300 divided by 30 is 10, did you prove the statement?

Chen: No.

Stavros: Why not?

Chen: Because you only picked examples that work! So, 3 divided by 2 doesn't work because 1.5 is not an integer.

Stavros: Exactly. You found a counterexample to disprove the statement, 'Dividing two integers creates an integer.' So, to prove the statement, 'If x and y are even integers, then $x + y$ is an even integer' using examples, you'd have to show that every possible example works—which you can't do because there are infinitely many even numbers.

Chen: Okay I get it. What do we do?

Stavros: What do you know about even integers?

Chen: They are multiples of 2.

Stavros: [The expressions are written on the whiteboard feature of Microsoft Teams. This feature is also available to Chen.]. Let 2m and 2n represent two arbitrary even numbers, where m and n are any integers. Now what?

Chen [writes]: 2m + 2n = 2(m + n). Okay I get it now. The answer is a multiple of 2 no matter what integers m and n are.

(Field note, October 2021)

This field note demonstrates Chen's understanding of the role examples play in proving and disproving statements using properties of integers. This nuance of understanding is foundational to the need to develop the formal agreed-upon structures of mathematical proofs to solve problems.

Stavros understood Chen's affinity towards applying theory to develop new ways to think about more generalized and abstract statements. Chen's engagement to the conversation confirmed the importance of designing dynamic curriculum that responds to the needs of a child through educative experiences (Dewey, 1938).

Chen drew our attention to the need for responsive curriculum making that considers and accounts for the life experiences of children. This example exceeds what teachers might reasonably do in classroom contexts, and we do not suggest such a level of differentiation for all children in all contexts. What we uncovered in this research is how an individual support person was able to facilitate a responsive curriculum. We know that mathematics teaching can often be fraught in terms of differentiation to accommodate high-level mathematics teaching (Ekstam et al., 2017; Prast, 2018). We came to see that the interactions between the child and Stavros had a significant impact on the child's identity making. One of the calls for NI is the social significance of an inquiry.

A Proof by Contradiction: Chen's Developing Flexible Thinking

Chen used examples, statements, and properties about integers that he understood fully as the mechanism to explore the abstract concepts of proving the statements. To prove statement P is true using the method of contradiction, the strategy is to assume P is false and then show this assumption leads to something false or impossible. Careful scaffolding is needed to overcome the pedagogical challenges fraught in this method, such as begging the question or relying heavily on concrete examples (Ko & Rose, 2022; Stavrou, 2014).

Below is a field text that informed our result about Chen's understanding of a *proof by contradiction* using a canonical example from number theory.

> *Stavros: Any real number belongs to one of two disjoint sets. What are the sets?*
> *Chen: The set of rational numbers or the set of irrational numbers.*
> *Stavros: Yes. If we choose any random number from the set of rational numbers, what form does it have?*

Chen: It's an integer or a fraction.

Stavros: Good. And don't forget that any integer can be written as a fraction by writing 1 in the denominator. Give me an example of an irrational number.

Chen: The square root of 2.

Stavros: Good. Let's use a proof by contradiction to prove the statement, 'The square root of 2 is irrational.' To start this, we are going to assume the opposite: the square root of 2 is rational. What does this mean?

Chen: It means the square root of 2 is a fraction.

Stavros wrote: $\sqrt{2} = \frac{n}{m}$, *where* n *and* m *are positive integers. Let's assume* m *isn't 0 and the fraction is reduced so that they don't share any common factors.*

Chen: Okay, because we can't divide by 0.

Stavros: To move to the next step, eliminate the square root.

Chen: Okay, so I have to square both sides:

$$2 = \frac{n^2}{m^2}.$$

Stavros: Now cross multiply.

$$2m^2 = n^2$$

Stavros: Is n^2 *even or odd?*

Chen: Even because it is a multiple of 2.

Stavros: Good. What about n?

Chen: Hmm... n *has to be even also because an odd number multiplied to itself is odd; but an even number multiplied to itself is even.*

Stavros: Good. We can rewrite n *as* $n = 2p$, *where* p *is a non-zero, positive integer. Substitute this into the equation you created and play around with it.*

Chen:

$$2m^2 = (2p)^2$$
$$2m^2 = 4p^2$$
$$m^2 = 2p^2$$

Chen: OH! That means m *is also even! OH! OH! But that's wrong!*

Stavros: Why is it wrong?

Chen: We said $\frac{n}{m}$ *is simplified—they can't BOTH be even at the same time.*

> *Stavros: Good! We reached a contradiction when we pretended that the square root of 2 is rational. Therefore, it must be irrational.*

(Field text, October 2021.)

Chen can flexibly work through problems using symbols and variables without getting confused by the abstraction. Stavros knew this, which helped scaffold Chen through the proof. Proving the irrationality of $\sqrt{2}$ is standard in post-secondary algebra courses. Stavros recalled learning the proof for the first time as an undergraduate student in a third-year number theory course. Stavros felt ill-prepared to deal with proofs because the foundational concepts were not appropriately scaffolded, leading to issues integrating proofs across various topics when his instructors taught advanced mathematics operating under the assumptions that students had the requisite knowledge. Stavros stated, "I encountered unnecessary struggles and barriers in my undergraduate and graduate courses because of this. I recall telling my master's supervisor that our undergraduate program requires a discrete math course for examining proofs. I was ignored, but such a course now exists." (Field note, December 2021).

This was a narrative thread in the stories of Stavros's experiences learning post-secondary mathematics that made him more attuned to instructional strategies that explicitly model proof construction. Stavros makes clear that the relational understanding of NI implies that the experiences of the researcher shape both the unpacking and experience of the research (Clandinin & Connelly, 2000).

Proofs by Induction: Stavros's Miseducative Experiences = Chen's Educative Experiences

A proof by induction verifies that a statement holds true for natural numbers. The first step of induction is the base case, where the statement is shown to be true for an initial value, such as $n = 1$. Next, the inductive step assumes the statement holds for $n = k > 1$ (called the inductive hypothesis) and then shows it also holds for $n = k + 1$.

Research shows that students often fail to recognize the importance of the base case; confuse the inductive hypothesis and the statement to be proven, leading to circular reasoning; rely excessively on empirical data; and struggle with the symbolic notation and algebraic manipulations during the inductive step (Michaelson, 2008; Norton et al., 2023). Stavros's planned thoughts reflected his own struggles learning proofs by induction, which shaped the statements he chose to present to Chen. "In my first year of university, I learned to prove the statement, 'For any natural number n, the sum of the first n natural numbers is equal to $\frac{n(n+1)}{2}$. I remember doing countless examples until I was convinced it worked; but, for me, the proof did not explain *why* it worked, so proofs by induction often didn't feel intuitive." (Field text, October 2021). Stavros relied on the empirical data of examples to support his understanding, but he never truly felt he made the conceptual leap to understanding the connection between mathematical induction and the result.

Reflecting on his own miseducative experiences guided Stavros to present Chen with intuitive, explanatory proofs by induction, such as the one used to verify the Heavy Coin Problem, a classic logic puzzle where you must identify a counterfeit coin. The task involves 3^n identical-looking coins, one of which is slightly heavier, with the challenge to identify the heavier coin using no more than n weighings on a balance scale. Stavros also chose this problem because Chen has an affinity to working with multiples, exponents, and large numbers. His conversation with Chen is summarized in the following first-person narrative:

> *"Chen, I need your help with a very important problem," I said as soon as we connected through Teams.*
>
> *"Okay, I guess I can try," Chen replied, beaming through his mask. "What's the very important problem?"*
>
> *"Did you remember to bring your chips? Pretend they are like my coins." I showed him a handful of plastic coins I had purchased last week.*
>
> *"Okay. I can pretend. It would be easier if you could come here," Chen said, referring to Covid precautions.*
>
> *I shared a fictitious story about being a coin collector and needing his help determining an efficient way to locate counterfeit coins by comparing their weights. I had an interactive balance scale shared on our screens. "If I have 3 coins, can I determine the counterfeit coin by using one weighing?"*

Chen quickly realized how one weighing would determine the counterfeit coin: place one coin on the left side of the scale, one coin on the right side, and leave the third coin on the table. If the scale is balanced, the coin not weighed is heavier; otherwise, the scale reveals the heavier coin. This demonstrated the base case: $n = 1$.

I applauded him for understanding that not every coin needs to be directly weighed to determine the heavier coin, hoping it would prompt him to apply this strategy when I added 6 more coins to the pile (i.e., $n = 2$ *involves* $3^2 = 9$ *coins). I asked, "Can we find the heavier coin in 2 weighings?"*

Chen shuffled coins around, murmuring ideas softly, his voice muffled by the mask he wore. Eventually, he stacked 3 coins on the left side of the scale, 3 on the right side, and 3 on the table. He giggled—which I knew to mean he had an aha! *moment—and said, "I know which pile has the heavier coin. Okay, so let's just pretend it's in the pile of coins on the table and that's why the scales balance. Then I can take the 3 coins from the table and split them up the way we started."*

Now I was the giddy one. I asked, "Let's pretend the number of coins I have is 3^k*, where* k *represents any number bigger than 1. What if* $k = 168$*? That's a big number because—"*

"Well, 168 isn't that big but 3^{168} *is more atoms than the whole universe."*

"What? How do you know that?"

"Well. You told me there are, like, 10^{80} *atoms in the universe."*

(Field note, October 2021)

This field note demonstrates the complexities of planned thoughts and action thoughts: Stavros's plan to teach induction proofs involved spontaneous decisions to discuss tangential ideas that Chen considered. If Stavros had responded to these movements away from the central problem as if they are disruptions to the main task, he will not have noticed that Chen was making comparisons with what he already knew by integrating prior knowledge. Furthermore, these brief digressions to examples and prior knowledge are strategies to manage the cognitive load and improve problem-solving performance (Gupta & Zheng, 2020).

The conversation turned into Stavros having more coins than atoms—an absurdity that made Chen laugh. Eventually, they returned to the proof. Building off the base case, using examples ($k = 1, 2, 3$) as empirical evidence, and utilizing prior knowledge about multiples of 3 allowed Chen to see the crux of the problem: he could always locate the pile with the heavier coin on the first weighing by creating 3 equal piles—one on the left of the scale, one on the right, and one on the table—and then iteratively creating 3 equal piles until there are 3 coins left. This demonstrated the inductive assumption: when there are 3^k coins, we can assume it takes k weighings to locate the heavier coin.

Stavros asked, "If I give you 3 more coins, what can you conclude?"

"It would take one more weighing because I can split them evenly on my 3 piles," Chen answered after a brief pause.

"This is the inductive step: we can weigh 3^{k+1} coins $k + 1$-times to locate the heavier coin. This completes the proof that if we have 3^n identical-looking coins, one of which is slightly heavier, we can locate it in no more than n weighings." (Field note, October 2021)

Discussion

Concluding Reflections: Planning Thoughts and Action Thoughts

Planning thoughts are integral to preparing for activities by reflecting upon what works and doesn't. Time constraints affecting preparation and activities that fail to captivate both mean sessions might have a trial-and-error dimension. Stavros's expertise in math and education provided him the competence (Creecy, 2020) to deliver specialized topics that Chen would otherwise not encounter until high school or beyond.

We recommend that educators approach planning thoughts as a process of posing questions with loose yet intentional goals, rather than setting rigid objectives that may lead to mis-educative experiences (Dewey, 1938). Stavros admitted to sometimes forcing activities because he already consumed varying amounts of precious time and did not want to see it go to waste. Flexible planning thoughts is a recursive approach to adapting to student needs and maintaining a coherent trajectory—or embracing inevitable decoherence and then pivoting.

We recommend educators develop capacities for action thoughts so that they are better prepared for the spontaneous decisions and responses inherent in complex classroom interactions (Coleman, 2014). Keeping in mind our role to co-compose curriculum with Chen in ways that support his identity making as a mathematician (Stavrou & Murphy, 2024), Stavros facilitated sessions by learning Chen's behavioural cues and responding appropriately to various in-the-moment instances that might undermine the progress of the session.

Conclusion

Effective teaching with precocious students involves a nuanced interplay of planned and spontaneous thinking. In terms of identity making, the result of this was a series of interactions in which Chen developed rigorous mathematical skills and competencies that are integral to mathematicians endeavoring to verify statements using the agreed-upon logic system of mathematical proofs. As educators, we must balance the expectations of the mandated curriculum with the creative conditions of constructivism that support the thinking processes and needs of a child whose life is in the making (Huber et al., 2011; Zain et al., 2012).

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