Canadian Journal for New Scholars in Education/ Volume 2, Issue 1
Revue canadienne des jeunes chercheures et chercheurs en éducation July 2009

Introducing Angles in Grade Four: A Realistic Approach Based on the van Hiele Model

Angela M. Smart

University of Ottawa
asmar022@uottawa.ca

Abstract

This article presents a discussion of the findings of a study of teaching and learning angles, following a theoretical framework that blends Realistic Mathematics Education (RME) and the van Hiele Model of Geometric Thinking. These theories also informed the design, experimentation, and evaluation of an introductory lesson on the idea of angle, and the relations between angles of different sizes in two different fourth grade classrooms. The teachers for the respective classrooms taught lessons that were based on a script and materials prepared by the researcher. At the end of the lesson the students provided written responses to questions on what they had learned during the lesson. Their answers and classroom observations provided a basis for the evaluation of the experiment. The research findings suggest the usefulness of lesson plans combining RME and the van Hiele Model in helping students develop an analytical conceptualization of angles.

Introduction

Previous research has shown that students tend to struggle with angle conceptualization (Battista, 2007; Clements & Battista, 1992), due to the abstract nature of angles and the multiple contexts in which angles can be represented. This article presents research examining students' conceptualization of angles. The goal of the study was to determine if a particular teaching method could be successful in helping Grade 4 students conceptualize the notion of angle. To that end, a lesson plan was designed using the teaching framework of Realistic Mathematics Education (RME), (Freudenthal, 1973; van den Heuvel–Panhuizen, 2001) the learning framework of the van Hiele Model of Geometric Learning, (National Council of Teachers of Mathematics [NCTM], 1988; van Hiele, 1986) and Euclid's definition of an angle (as cited in Sibley, 1998, p.287). The overarching aim of this study is to gain insight into whether a lesson taught according to RME teaching theory can produce an analytical conceptualization of angles amongst Grade 4 students at one of the first three levels of the van Hiele Model.

Theoretical Framework

This research is informed by the theories of RME (Freudenthal, 1973; van den Heuvel–Panhuizen, 2001) and the van Hiele Model of Geometric Thinking (NCTM, 1988; van Hiele, 1986). The following section describes RME and the van Hiele Model. A later discussion will explain how they were incorporated into the lesson plan.

Realistic Mathematics Education (RME)

Traditional mathematics education is sometimes referred to as mechanistic mathematics education (van den Heuvel-Panhuizen, 2001, p. 1). Students are introduced to mathematics as though it were a ready–made system of symbols and solutions, wherein they mechanically drill problems by mimicking the procedure the teacher demonstrates. Thus, mathematics becomes a very mechanical process in which a true understanding might never really be achieved. RME is a Dutch response to such traditional mathematics education techniques. Founded from the work of Hans Freudenthal (1973), RME places an emphasis on making mathematics education a relatable, usable experience for students. This is achieved by emphasizing the connections between the real world and the mathematical world. By staying close to reality, educators are able to give mathematics a sense of value in which students can find real life associations (Freudenthal, 1973).

The teaching theory of RME is comprised of five main characteristics: (a) introducing a problem using a realistic context; (b) identifying the main objects of the problems and developing models of the problem with these objects; (c) using appropriate social interaction and teacher intervention to refine the models of the problem for an optimal representation for the problem; (d) encouraging the process of reinvention with the development of a model for the problem; and (e) focusing on the connections and macro–didactic aspects of mathematics in general (van den Heuvel–Panhuizen, 2001). According to RME, these characteristics must be enacted together in order for a topic to be taught appropriately (Freudenthal, 1973).

The van Hiele Model of Geometric Thinking

Central to the van Hiele Model is the concept that there are five levels or stages (Levels 0 to 4) of geometric thinking wherein students' understanding could reside (van Hiele, 1986). These levels are assumed to be sequential. In order to ensure success at a higher level, a student should be successful at all of the previous levels. Van Hiele believed that if teachers were able to understand the five levels of geometric thinking, they would be able to assess the level at which their students understand and adjust their teaching accordingly. The basic idea underlying the van Hiele levels is that each higher level focuses on properties of the objects of attention of the previous level. As well, the van Hiele Model allows educators to develop lesson plans that move through the different levels sequentially. The van Hiele Model provides researchers with a framework in which to study students' learning steps and their conceptualizing of geometry topics (NCTM, 1988; van Hiele, 1986).

Level 0 is the visual level in the van Hiele Model. Conceptualization at this level includes being able to identify, name, and reproduce geometric objects by visual recognition. Level 1 represents the level of properties and descriptions. Conceptualization at this level includes identifying or describing geometric objects according to their properties. Level 2 is the level of pre–deduction or relationships. Conceptualization at this level comprises being able to identify relationships between the properties of a geometric object as well as relationships between different geometric objects. Level 3 is the level of deduction. Conceptualization at this level is connected to being able to deductively prove the relationships identified at Level 2. Level 4 is considered the level of rigor. Conceptualization at this level involves being able to rigorously analyze the validity of the deductive system developed in Level 3.

Literature Review

Several literary strands in the areas of mathematics, geometry, teaching, learning, and conceptualization inform this study. In particular, this investigation is guided by various angle definitions and studies on angle conceptualization in combination with the theoretical framework of RME and the van Hiele Model that shaped this research. This section provides a review of the main literary resources on angles that have been employed within this inquiry.

Angle Conceptualization

There is an ample body of research that has been conducted on the topic of angle conceptualization over the last two decades, as described in the Mathematics Education Handbooks (Battista, 2007; Clements & Battista, 1992). One focus of angle conceptualization research in the late 80s and early 90s has been on the use of angle–making computer software, particularly *Logo*. Clement and Battista investigated whether instructing students in *Logo* could help students conceptualize angles, shapes, and motion (1989) and help with the conceptualization of the notion of angle and the angles of polygons (1990). Another study of *Logo* was Hillel and Kieran's (1987) exploration of how students chose their inputs when instructing a turtle to make an angle.

As research into the use of *Logo* lessened, researchers began investigating students' conceptualizations of angles in realistic physical contexts. For example, Mitchelmore (1998) examined how students in Grades 2, 4, and 6 conceptualize an angle as a rotation or a turn by examining three-dimensional angle-making models. In a follow-up of this research, still using physical models, Mitchelmore and White (2000) investigated whether students could move from a conceptualization of angles as turnings or rotations to an abstract conceptualization of angles in general. In more recent studies, the use of physical three–dimensional models has moved towards a trend of embodiment. For example, Fyhn (2008) took a class of Grade 8 students to a climbing wall for a day to help them understand angles through their physical motions and positions. Yet even with all of this work and different approaches, the research suggested that students still struggle with angle conceptualization.

Angle Definition

One of the challenges to teaching angles is that there has been no single definition of the notion of an angle (Brown, Simon, & Snader, 1970; Hartshorne, 1997; Sibley, 1998; Sobel, Maletsky, Golden, Lerner, & Cohen, 1986; Webster Dictionary & Thesaurus, 2002). A review of angle definitions shows three themes or categories emerging. In particular, an angle can be categorically defined as either a geometric shape, a measure, or a dynamic rotation (Henderson & Tainmina, 2005). An example of an angle definition that can be categorized as a geometric shape is that of Hilbert (as cited in Sibley, 1998), who claimed: "By an angle is meant a point (called the vertex of the angle) and two rays (called the sides of the angle) emanating from the point" (p. 294). Webster's Dictionary and Thesaurus (2002) presents an example of an angle as a measure: "A measure of an angle or the amount of turning necessary to bring one line or plane into coincidence with or parallel to another" (pp.37-38). Lastly, Mitchelmore (1998) gave an example of an angle that is defined by dynamic rotation: "[an angle is] the amount of turning between two lines about a common point" (p. 265). To summarize, an angle will belong to one definition category or another based on the context within which it is presented. Thus, it can be said that to understand a particular definition of an angle, students must also understand the particular context situation in which that notion of an angle is represented.

Euclid's angle definition was the one chosen for this study because of its fit with the particular frameworks of Realistic Mathematics Education (RME) and the van Hiele Model. Sibley (1998) offers Euclid's angle definition as: "A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line" (p.287). This definition successfully conveys the physical and visual properties of an angle in accordance with the first van Hiele Level (NCTM, 1988). This study includes a lesson plan that introduces the notion of a flat angle. Since Euclid's definition specifically states that angles do not "...lie in a straight line" (as cited in Sibley, 1998, p.287), in the lesson plan, flat angles are represented as the result of adding angles together that produce a straight line. Overall, the lesson that is discussed in detail below, follows Euclid's definition while focusing on the introduction of angles by recognizing the inclination between two lines.

Methodology

The study that is discussed within this paper is comprised of the examination of the effects of a lesson plan on angle conceptualization. The lesson plan that was used in this study (see Appendix A) consisted of five activities that were carried out sequentially within a Grade 4 classroom setting. Two Grade 4 teachers used the lesson plan in their classrooms. Class 1 had 25 students while Class 2 had 26 students. Both of the classroom teachers worked at the same local public elementary school. The lessons were taught on different days, one week apart from each other, so the researcher could attend and observe the lessons.

Prior to the planned lesson, the researcher met with the teachers individually and provided them with a lesson plan package. This package included a copy of the lesson plan (Appendix A), a set of teaching transparencies (Appendix C), a set of blank transparencies, and one student workbook (Appendix B). At that time, the researcher instructed the teachers to read through the lesson plan and attempt the activities on their own before the date that the lesson would take place. Each teacher met with the researcher again before the lesson in person to answer any pre-lesson questions.

The lessons took place during the usual math time in the regular classrooms, and they were taught by the students' classroom teachers. A key part of this inquiry involved maintaining the regular classroom teachers for instruction during the research lesson and positioning the planned lesson during the regular time allotted for the study of math. The maintenance of a routine for engagement in math education with the students was central for an interpretation of

RME that was adopted for this investigation that focused on the provision of an appropriate and realistic context situation for math study.

On the days that the lesson took place the researcher arrived with the materials for the students, including a student workbook; a set of blank transparencies, and a transparency pen for each student. In both classrooms, the regular math lesson occurred after a recess break and lasted for approximately one hour. During recess, the aforementioned materials were arranged on each student's desk. As well, a small digital voice recorder was placed in a discrete location in the classroom to audio–record each observed lesson as it took place. The researcher observed the lessons from the back of the classrooms so as not to interfere with the students but could accurately see what the teacher was showing on the board. Observations were recorded in a notebook. At the end of the lesson, the researcher collected the students' workbooks that contained written responses to questions from the final planned activity. At the end of the lesson, the researcher also collected the transparencies that the students used throughout the lesson's activities.

Sequential Lesson Plan Activities

In this sub-section, the main points of each of the five sequential activities are discussed. Details for each activity are outlined, and reference is made to connections with theories about angles and angle conceptualization. Furthermore, a rationale is given for each lesson plan activity with respect to steps and stages in RME and/or the van Hiele method.

Activity 1 was designed to introduce angles to students in a way that was consistent with the RME theory of using realistic context situations and with Euclid's definition of an angle. Using transparencies, the concept of angles and right angles were introduced. To begin, two roads were shown to the students. One road had a turn, and thus, formed an angle, while one road did not. The road with the turn was presented with two lines that had an inclination to one another. Roads were also shown that intersected at different angles in order to explore the idea of identifying right angles with the students.

This activity also contained an exploratory exercise for students. Using the first page of the student workbook and a blank transparency, students were asked to copy the angles they saw in the picture on the first page of their student workbooks onto a blank transparency sheet and to identify any right angles. The process of identifying, recognizing, finding, and reproducing angles was consistent with teaching the topic of angles beginning with Level 0 of the van Hiele Model of Geometric Thinking. This process was also consistent with the RME characteristic of rediscovery, as the students discover and copy the angles for themselves.

The next planned activity for the observed lesson was designed to help solidify students' visual understanding of angles to ensure that they were working at or within Level 0 of the van Hiele Model. The activity also continued to use realistic context situations and rediscovery, consistent with RME. Using blank transparencies, students identified and copied angles from pictures in their student handbooks that depicted mountains and a pair of scissors.

Following that activity, the students participated in an activity that was designed to introduce the notion of a flat angle and the idea of adding angles to make different angles. The teacher placed two right angles that were drawn on transparency sheets on the overhead projector and slid the images together to make a flat angle. Once this was demonstrated to the students on the overhead, the teacher asked the students to stand up and work in pairs on a physical exercise. Each student needed to use hers or his arms to make an angle that would produce a flat angle when added together with a partner's angle. This activity allowed for a level of social interaction as outlined in RME, because the students investigated angles with their peers. Figure 1 was provided for the teachers during lesson planning to clarify the body action of this activity.

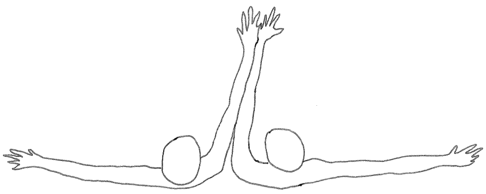


Figure 1: Embodied angles

The activity took place in Level 0 of the van Hiele Model in that it introduced students to a visual conception of a flat angle. It also represents Level 1 of the van Hiele Model in that it introduced the concept of adding angles together to produce different angles, which is a property of angles. The activity is important because it provided a realistic context situation, namely the students' own bodies, and it enabled students to embody the notion of angles and give tangible form to something abstract (Freudenhal, 1973).

Activity 4 was the last instructional activity of the lesson plan. This activity was designed to see if students could use the knowledge that was taught in the previous activities to re/discover a specific property about angles. This activity required students to draw an arbitrary triangle and an arbitrary quadrilateral inside their workbooks. Then, using pieces of blank transparency, the students were instructed to copy the angles from their shapes and add them up to see what happens. This activity assumes that students have a good understanding of angle at Level 0 of the van Hiele Model. It is also consistent with the teaching theory of RME in that it promotes a hands–on activity that will allow the students to re/discover significant properties for themselves.

The final activity for the studied lesson was designed to inform the teachers and the researchers about the students' levels of understanding. For this activity, the students were asked to answer two questions in complete sentences in their notebooks. The questions were: "What was math class about today?" and "What did you learn?"

Data Analysis

The data for this study were collected from a variety of methods. They included the students' workbook questions, the lesson transparencies, the field notes taken during the in–class observation, and the audio–recorded transcriptions made in each classroom. The data were analyzed as described below.

Written Student Responses

The students' written responses to the two questions at the end of their workbooks were coded according to content and categorized as either narrative or analytical, following an approach adapted from Fyhn (2008). Narrative responses were those that did not contain any mathematical response or description. Instead, they told a story about what took place or expressed a personal opinion of the students. These were divided into N1 and N2. N1 indicates narrative responses that mentioned angles, while N2 denotes responses that did not mention angles.

Analytical responses contained words or phrases that described angles in a mathematical context. These were divided into A1, A2, A3, and A/E. A1 refers to analytical responses that described angles at the visual stage of van Hiele Level 0 (E.g. finding, recognizing, or, drawing). A2 analytical responses described angles at the descriptive stage of van Hiele Level 1. This included all responses that mentioned adding angles or putting angles together to form different angles. A3 analytical responses mentioned something of the pre–deductive nature in regards to angles at the informal deduction stage of van Hiele Level 2. Only responses that dealt with Activity 4 fell into this category. A/E was a code used to describe analytical responses with error. In addition, some responses required a different category, because they did not fit into the above analysis. These were responses that were left blank, were written illegibly, or that were incomplete. Such responses were categorized as N/A.

*Student Transparencie*s

The students' transparencies were analyzed according to whether or not the students drew their angles correctly and whether or not the students included an arc or square (for right angles) in the corner. An analysis of arcs and squares was included, because it was interesting to see how many students integrated this component to complete the drawing of their angles. Drawing an arc was not a component of the lesson plan, but both teachers, either informally or formally, instructed students to use arcs or squares in the corners of their angles.

In–Class Observations and Audio–Recording Transcriptions

The in–class visits were analyzed according to prominent observations. While no categorizations were assigned, records were maintained regarding teachers' deviations from the prescribed lesson plan. Classroom observation data were also analyzed according to notations of the teachers' methods of teaching, such as via open–ended questioning or class participation. Furthermore, general observations were documented about how the students responded to the lesson, and unexpected events or behaviors. These observations were compared with the audio transcriptions for analysis purposes.

Findings

The findings of this study are presented in terms of the written responses, the study transparencies, and the in–class observations and audio–recording transcriptions. Significant findings are noted for the individual classes. Comparisons are also made for the findings between classes in terms of similarities and differences.

Written Responses

The categorizations of the students' responses to the workbook questions are presented below in Tables 1A and 1B. The numbers and percentages in the table represent the portion of student responses of a specific coding.

Table 1A: *Class 1 Written Responses*

Coding	Q1	Q2
A1	11 (44%)	3 (12%)
A2	6 (24%)	11 (44%)
A3	1 (4%)	0
N1	3 (12%)	1 (4%)
N2	1 (4%)	5 (20%)
AE	2 (8%)	3 (12%)
N/A	1 (4%)	2 (8%)

Table 1B: *Class 2 Written Responses*

Coding	Q1	Q2
A1	15 (57.8%)	10 (38.5%)
A2	1 (3.8%)	9 (34.6%
A3	1 (3.8%)	1 (3.8%)
N1	7 (27%)	3 (11.6%)
N2	1 (3.8%)	1 (3.8%)
AE	1 (3.8%)	0
N/A	0	2 (7.7%)

In Class 1, a total of 20 students out of 25, or 80%, answered at least one of the questions analytically. Twelve students provided analytic answers to both questions. Only one student made an A/E (error) response on both questions. Many more students gave A2 responses for Question 2 than A1 responses.

In Class 2, a total of 21 students out of 26, or 81%, answered analytically to at least one of the questions. Sixteen students answered analytically to both. Only one A/E response was made overall. One student answered both questions at an A3 level. Classroom 2 also had fewer N2, AE, and N/A responses than Classroom 1.

Transparencies

Tables 2 through 5 represent the results collected from the students' transparencies. However, the transparencies also presented some results that are not represented in the figures.

Table 2:*Transparency Results for Scissors Activity*

	Angles with arcs	Some angles with arcs	Angles without arcs	Blank	Angles drawn incorrectly	Missing
Class 1	20	2	1	0	2	0
Class 2	9	0	13	1	2	1

For the scissor picture, two students in each class drew some of their angles incorrectly. In Class 1, one student drew lines on the sides of each of the legs of the scissors but did not connect the lines at the tip of the scissors, thereby excluding what is actually the angle. The other student copied the round shape from the scissors' handles and identified those as angles. In Class 2, both of the students that drew the angles incorrectly used more than two rays.

Table 3: *Transparency Results for Roadmap Activity*

	Angles with arcs	Some angles with arcs	Angles without arcs
Class 1	24	1	0
Class 2	14	1	11

With the Roadmap, only one student in Class 1 drew fewer than 20 angles. This student drew only seven angles. In Class 2, the students who drew their angles with no arcs on the Roadmap had a variety of different ways in which they labeled their right angles. Two students marked their right angles with a dot, which is something that the teacher did in front of the entire class. Eight students marked their right angles with an "R". One student coloured in a complete triangle in the corner of the angle for each of the right angles.

Table 4: *Transparency Results for Mountain Activity*

	Angles with arcs	Some angles with arcs	Angles without arcs
Class 1	18	5	2
Class 2	7	1	18

Table 5: *Transparency Results for Activity 4*

	Angles with arcs	Some angles with arcs	Angles without arcs	Angles drawn incorrectly	Missing
Class 1	18	3	3	0	1
Class 2	21	2	1	2	0

In Activity 4, one set of the transparencies that were drawn incorrectly had the edges of the shapes copied but the edges did not meet up with the corner, while the other set of transparencies had coloured in triangles for the angles with no rays.

In–Class Observations and Audio–Recording Transcriptions

The researcher observed differences with both the teachers and the students in the two classroom situations. The teachers expressed differences in their attitudes toward undertaking the planned lesson on angles. Teacher 2 was excited about the lesson plan, informed me that she was impressed with it, and wanted to know more about my research and whether the researcher had developed other lesson plans she could try in her classroom. Teacher 1 did not mimic these opinions or feelings, but expressed that it was "just another lesson."

The teachers were also different in terms of their stages in their professional careers. Teacher 1 had already completed her teaching career, and had actually returned from retirement to teach for one more year at the school's request. Teacher 2 was near the beginning of her career as a teacher.

At the same time, the teachers displayed variations in their teaching methods for the research lesson. During the lesson plan, Teacher 1 left out a number of the instructional transparencies and changed the order of some of the instructions. Teacher 2 used all of the transparencies during the lesson and followed the order that the lesson plan directed. Teacher 1 instructed the students to find as many angles as they could in the roadmap, mountains, and scissor pictures, which caused a time constraint near the end of the lesson. Thus, Activity 4 was very rushed. Teacher 2 limited the number of angles the students were to find, which left more time for the other activities. Teacher 1 started but did not complete the activity where the students form angles with their arms, which was a component of Activity 3. Teacher 2 fully completed this activity with the class.

As well, Teacher 1 maintained a very high level of discipline in the classroom. She would not allow for any extra talking from the students while they were completing the activities. Teacher 2 was very lenient about student talking. She ignored the extra chatter that was going on when the students were working at their desks. As a result, there were variations within the audio–recorded transcriptions. The audio transcriptions that contained the teacher–student interchanges differed in length. There were 11 pages of text from Class 1 and seven pages of text from Class 2. This also reflects the amount of verbal instructions that was actually given to the students.

There were 50 recorded responses/questions made by students in Teacher 1's classroom, where some students responded more than once, and 23 recorded responses made by students in Teacher 2's classroom, with some students responding more than once. This was evidence that, within Class 1, there was more interaction between individual students and the teacher while the entire class was listening. On the other hand, the researcher observed that Teacher 2 spent much more time helping students at their desks while the students were completing the activity tasks. This was a different type of student–teacher interaction. Unfortunately, the audio recording did not catch the conversations between Teacher 2 and individual students because of background talking.

Discussion

The majority of the written responses to the activities in the lesson plan for this study were coded as analytical A1, A2, or A3 regardless of the classroom. This means that the students' answers described angles at either Level 0 or Level 1 of the van Hiele Model. In regards to the differences, more students in Class 1 may have answered analytically to both questions because of the observation that Teacher 1 instructed the students to be "very detailed" when filling in their answers. This comment may have led the students to feel they needed to respond in a narrative fashion, as though they were completing a narrative writing assignment.

A potential reason for the majority of A2 responses to Question 2 (What did you learn?) for Class 1 might be that the students had some previous experience with the concept of angle prior to the research lesson. Therefore, the knowledge that the students reported to have learned was more advanced information, like adding up angles. Unfortunately, the researcher did not test for previous knowledge prior to the lessons. Other researchers (Fyhn, 2008) have tested for prior knowledge before conducting their studies and found that students did report on the new knowledge obtained when questioned about what they learned.

When considering the A3 responses, the frequency reported from Class 1 and 2 is not different, even though Class 2 spent more time on the activity than did Class 1. As well, Teacher 1 did point out the results to Activity 4 to her class, while Teacher 2 specifically left out the results. It could be concluded that the student in Classroom 1 was repeating the information Teacher1 told the class. More students in Classroom 1 may not have responded at an A3 level because Teacher 1 rushed through Activity 4. There also does not seem to be any direct reason, even after reviewing the observation notes and the audio transcripts, for having fewer A/E responses in Classroom 2 than in Classroom 1.

Conclusion

The results of this study suggest that a majority of students in both classes gained an analytical understanding of angles within at least Level 0 of the van Hiele Model. This means that at a minimum, they comprehended a visual representation of angles, with some starting to develop an understanding of the properties of angles. The results of the study also imply that the differences in how the lesson plan was implemented in the different classroom settings did not interfere with the initial conceptualization of the notion of angles, because the results from the students were similar in both Class 1 and 2. Given the dramatic differences in how the lesson plan was implemented, this was a surprising result. Nevertheless, this study exemplifies the use of realistic context examples, such as within the activity instructions and the pictures of the roadmap, mountains, and scissors, combined with the use of transparencies to trace the angles, as a successful strategy for developing an initial conceptualization of angles amongst students. For this reason, the RME approach to teaching appears to be a promising method for introducing angles at this grade level.

Furthermore, proposed improvements to the lesson plan used in this study include adding instructions on how to draw or copy an angle within the first activity. This would hopefully assist the students who were observed as needing help with these processes. Classroom observations revealed that the lesson plan would also benefit from leaving out Activity 4, since students tended to struggle with adding up the angles in that activity. The study would therefore be improved with the inclusion of instructions on angle congruency prior to introducing the concept of angle addition. However, the findings of this study display the use of transparencies as a very successful way to teach angle congruency, because students would again be able to compare different angles by moving and manipulating them to coincide on top of each other.

Future research might be conducted on more advanced topics related to angle conceptualization. Lesson plans using the methods of RME and the transparencies may help students discover the different properties of angles, like angle congruency, which was mentioned above. As well, the concept of measuring angles is a natural next step after the conceptualization of angles. A lesson plan may be developed to teach angle measurement using transparencies that would allow students to compare and manipulate angles according to measurement size. There is also the potential to develop a series of lesson plans that followed the conceptualization of the notion of angle from its initial introduction to an understanding of angle measurement, as it is part of the Grade Four curriculum. In summary, the findings of this study support the notion that using realistic context problems and the teaching theory of Realistic Mathematics Education helps shape students' understanding of angles in the particular ways outlined by the van Hiele Model of Geometric Thinking.

References

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning: Vol. 2* (pp. 843-908). Charlotte, NC: Information Age.

Brown, K. E., Simon, L., & Snader, D. W. (1970). *Applying high school mathematics.* Toronto: Laidlaw Brothers.

Clements, D. H., & Battista, M. T. (1989). Learning of geometric concepts in a *Logo* environment. *Journal for Research in Mathematics Education, 20*(5), 450–467.

Clements, D. H., & Battista, M. T. (1990). The effects of *Logo* on children's conceptualization of angles and polygons. *Journal for Research in Mathematics Education, 21*(5), 356–371.

Clements, D., & Battista M., (1992). Geometry and spatial reasoning. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. New York: Macmillan Publishing.

Clements, D. H., & Burns, B. A. (2000). Students' development of strategies for turn and angle measure. *Educational Studies in Mathematics, 41*(1), 31–45.

Freudenthal, H. (1973). *Mathematics as an educational task.* Holland: D. Reidel Publishing Company.

Fuys, D., Geddes, D., & Tischler, R. (1988). The Van Hiele model of thinking in geometry among adolescents. Journal for Research in Mathematics Education. Monograph, Vol. 3, i-196.

Fyhn, A. B. (2008). A climbing class' reinvention of angles. *Educational Studies in Mathematics, 67,* 19–35.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics, 39,* 111–129.

Hartshorne, R. (1997). *Geometry: Euclid and beyond.* New York: Springer.

Henderson, D. W., & Tainmina, D. (2005). *Experiencing geometry: Euclidean and non–Euclidean with history* (3rd ed.). New Jersey: Pearson Education.

Hillel, J., & Kieran, C. (1987). Schemas used by 12–year–olds in solving selected turtle geometry task. *Recherches en Didactique des Mathematiques, 8*(1.2), 61–102.

Mitchelmore, M. C. (1998). Young students' concepts of turning and angle. *Cognition and Instruction, 16*(3), 265–284.

Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalization. *Educational Studies in Mathematics, 41,* 209–238.

National Council of Teachers of Mathematics. (1988). *The van Hiele Model of thinking in geometry among adolescents: Journal for Research in Mathematics Education, monograph number 3.* United States of America: Author.

Sibley, T. Q. (1998). *The geometric viewpoint: A survey of geometries.* Massachusetts: Addison Wesley Longman.

Sobel, M. A., Maletsky, E. M., Golden, B. N., Lerner, N., & Cohen, L. S. (1986). *Geometry.* New York: Scribner Educational Publishers.

van den Heuvel–Panhuizen, M. (2001, November). *Realistic mathematics education as work in progress.* Paper presented at the meeting of The Netherlands and Taiwan Conference on Mathematics Education, Taipei, Taiwan.

van Hiele, P. (1986). *Structure and insight: A theory of mathematics education.* Orlando, FL: Academic Press.

Webster Dictionary & Thesaurus of the English Language. (2002). *Angle (*pp. 37–38). Canada: Trident Press.

Appendix A

The Lesson Plan used for the In-class Experiments

Directions: Follow the activities in order listed.
Materials needed:
-Handout containing context pictures and worksheets for each student
-Set of blank transparencies for each student
-Non-permanent pen for each student
-Transparencies referred to throughout the lesson for teacher

Activity 1: Introducing angles and right angles as well as identifying them.

Place transparency #1a on the overhead
Refer to the first picture and tell the class "This road has a turn, or an angle."
Refer to the second picture and tell the students "This road does not."
Place transparency #1b onto #1a on the overhead
Refer to the first picture and tell the class "The red line and the blue line cross."
Refer to the second picture and tell the class "The red line and the blue line do not cross."

Place transparency #2a on the overhead
Refer to the first picture and tell the students "These roads cross at an angle."
Refer to the second picture and tell the students "These roads cross at a right angle; they are perpendicular."
Place transparency #2b onto #2a on the overhead
Refer to the first picture and tell the class "The lines cross at an angle."
Refer to the second picture and tell the class "The line cross at a right angle."

Place transparency #3 on the overhead
Refer to the first picture and tell the students "Where the lines cross the angles are not equal on both sides."
Refer to the second picture and tell the students "Where the lines cross the angles are equal on both sides. When the angles on equal on both sides, they are right angles."

Place transparency #4 (Road map) on the overhead
Ask the students to turn to his or her copy of the road map that is in the handout.
Tell the class "This road map is full of angles, some of them are right angles and some of them are not."
Using a clean transparency, ruler and non-permanent pen, demonstrate to the class how to copy an angle from the road map.
Ask the class to follow your example and copy a number of angles (maybe 10) from the road map onto the clean transparency labeled "Road Map". Ask the students to label the right angles in the angles they have copied.

Activity 2: Finding angles in context pictures.

Ask the students to look at the pictures on page 2 of the handout.
Tell the student "Each of these pictures has angles in them."
Ask the students to find the angles in the pictures and copy the angles onto the clean transparencies labeled "Scissors" and "Mountains".

Activity 3: Introduce the flat angle

Place transparencies #5a and #5b (the identical right angles) on the overhead. Make sure the right angles open in opposition directions and are on the same axis. (See diagram 1 for an example)

Tell the students “Two right angles make a flat angle.” While you tell them this, slide the two transparencies so that the two transparencies line up and form a t-intersection.
Refer to the new figure and tell the students “The arms of a flat angle lie in a straight line.”
Place transparencies #6a and #6b on the overhead
Like with the previous transparencies, slide these two together to line up.
Tell the students “If two angles are added together and make a straight line, their sum is a flat angle.”
Ask two students to stand up. Have both students make a right angle with their arms in opposition directions. Have the two students slide shoulder to shoulder to demonstrate that the two right angles make a flat angle. Then have the students hook pinky fingers on the hands that are held out in front and touching. Then demonstrate how the students can move those arms back and forth to form different sizes and angles that will still sum up to a flat angle. Get the rest of the class to stand up and try the activity with a partner. (See diagram 2 for an example)

While students are still standing, have two groups of two students each stand back to back and have the students with the outside arms hook pinky fingers. This demonstrates to the students that the flat angle is definitely a line.

Activity 4: Adding up other angles

Adding up a triangle’s angles: Have the students draw any type of triangle in the handout on the page labeled “triangle”. Make sure that each student makes a triangle that is different than his or her neighbor’s. Tell them to use three little pieces of transparency to copy the three different corner angles. Then have students add up the angles and see if they can see something special about the resulting added up angle. Ask the students to look at his neighbors work and see if there is something special about their added up angles. If they do not come to the conclusion that the angles add up to make a straight line (or some variation of that conclusion), then have the students repeat the exercise with a different triangle.

Adding up quadrilateral angles: Have students draw any four-sided shape (does not have to be a square or even have right angles) in the handout on the page labeled “quadrilateral”. Make sure that each student makes a quadrilateral that is different than his or her neighbor’s. Tell them to use four little pieces of transparency to copy the four corner angles. Then have the students add up the angles and see if they can see something special about the resulting added up angles. Ask the students to look at his neighbor’s work and see if there is something special about their added up angles. If they do not come to the conclusion that the angles add up to make a complete circle (or some variation of that conclusion), then have the student repeat the exercise with a different four-sided shape.

Activity 5: Journal entry

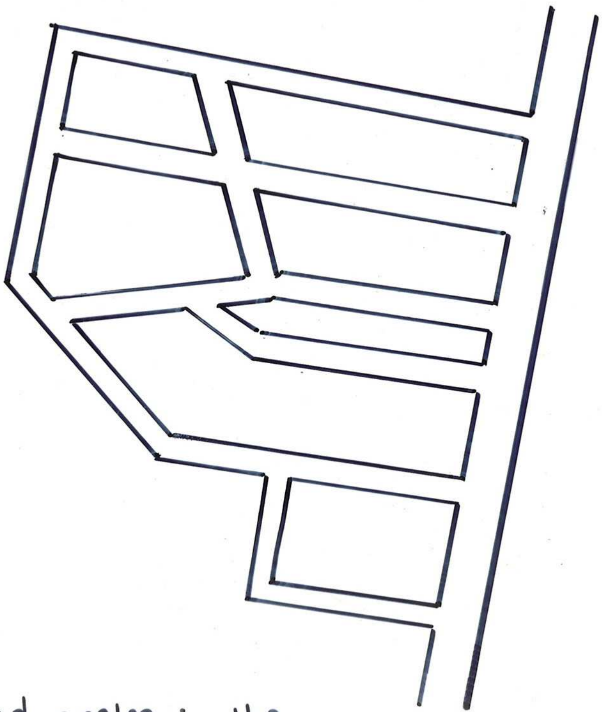
On the last page of the handout ask students to answer the two questions in complete sentences.

Appendix B

The Students' Workbook

#4

Road Map



- Find angles in the figure
- Label the angles that are right angles

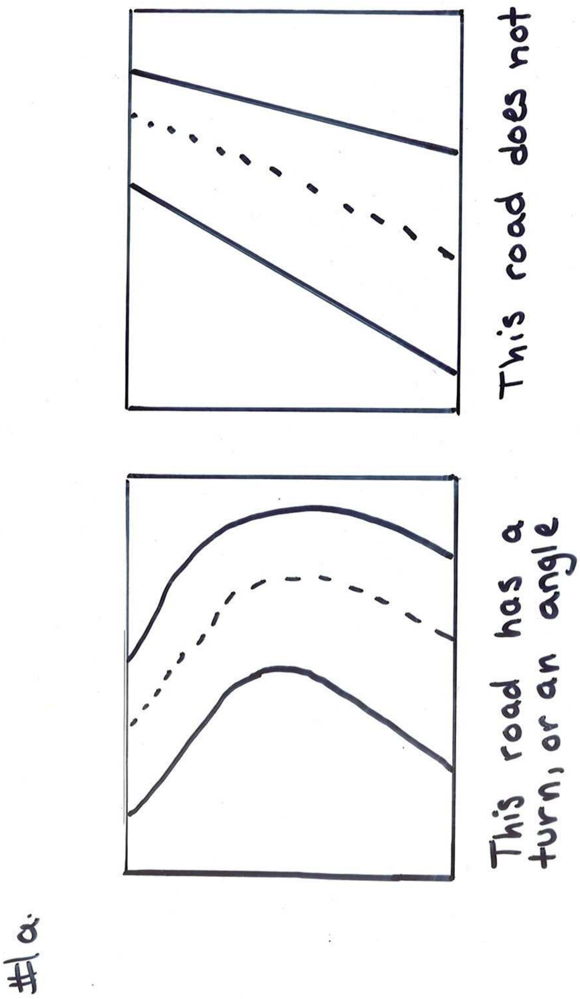
Triangle:

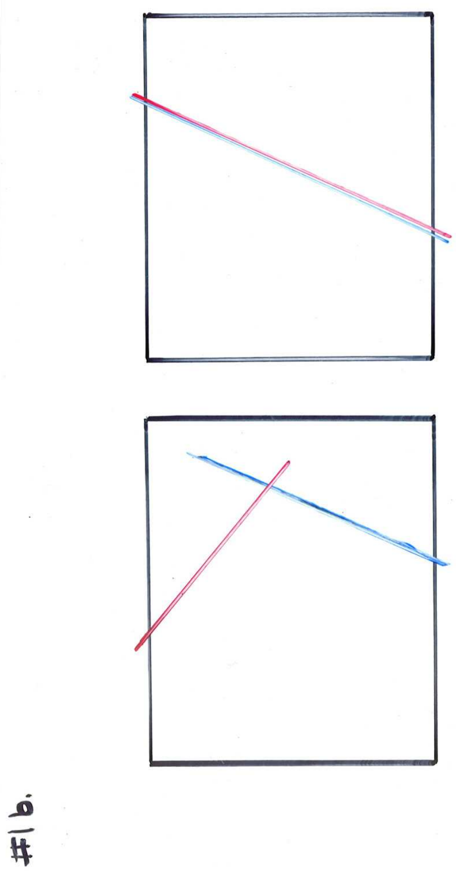
Quadrilateral (four-sided shape):

What was math class about today (answer in full sentences)?

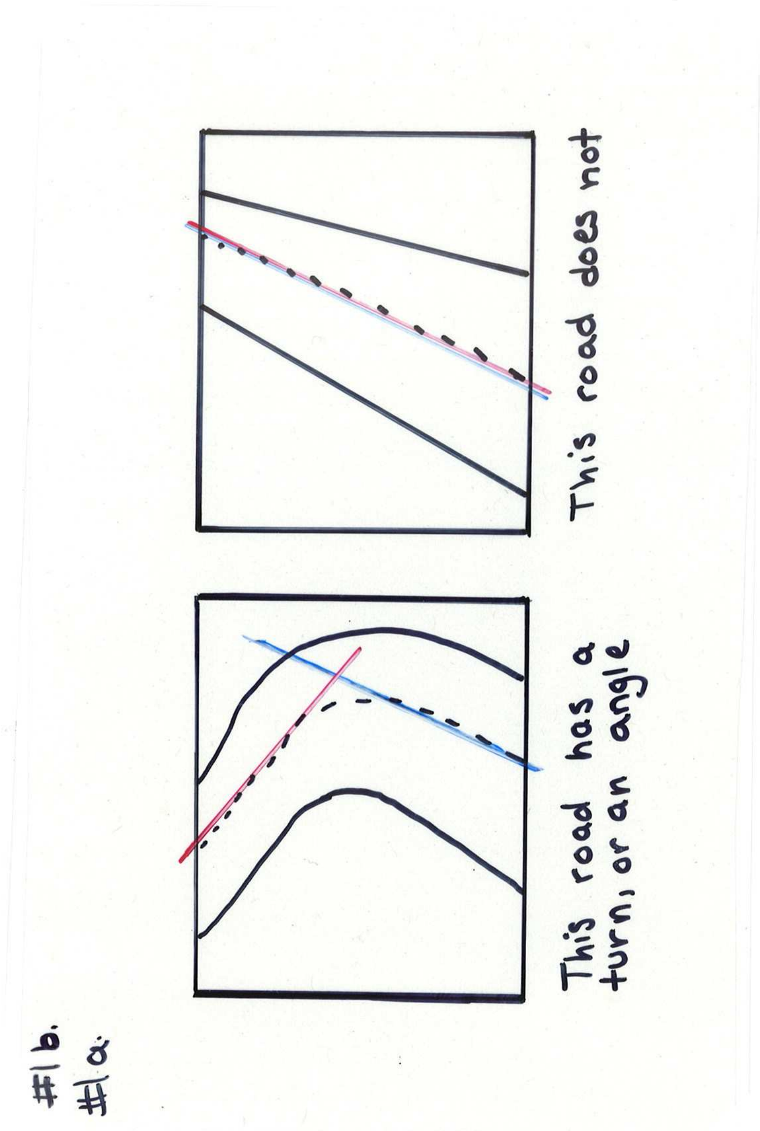
What did you learn (answer in full sentences)?

Appendix C
The Teaching Transparencies

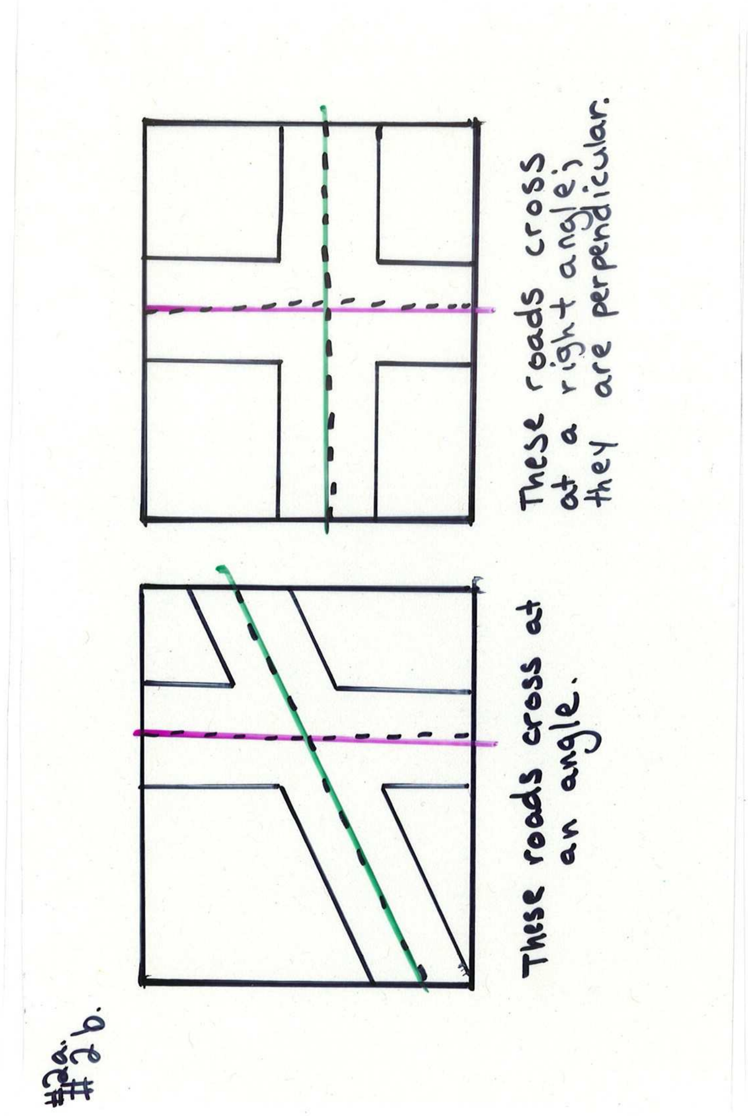



#6

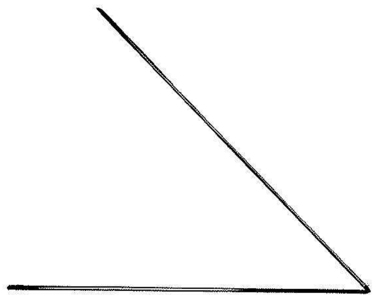
Transparencies #1a and #1b were meant to be shown together as in the figure



Transparencies #2a and #2b were meant to be shown together as in the figure below



#6a.



#6b.